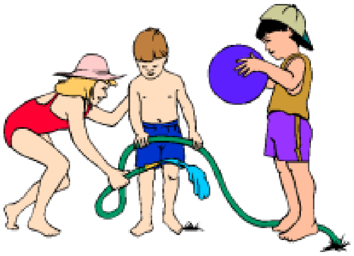
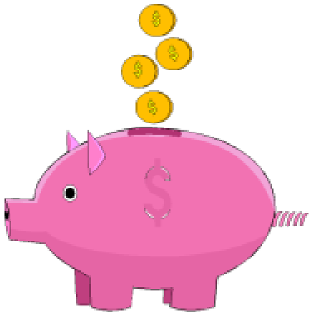
The Growth and Decay Formula

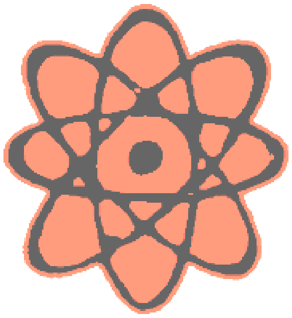
Many things grow and shrink (decay) at a rate based on how much stuff there is at the moment.



For instance, the number of births in a **population** is based on the number of people living at the time (since the more people there are, the more new people there will be).



Another example is **compound interest**: As money accumulates in the account (due to earned interest), the more interest the account will earn; that is, the interest will earn interest.



In science, we can observe the **radioactive decay** of an element like uranium. As the uranium decays, less of it will decay because as time goes on, there's less of it left to decay.

❑ THE GROWTH AND DECAY FORMULA

Here's a formula to help you predict how much of something there will be in the future. Among hundreds of applications, it works for business depreciation, the spread of disease, and the half-life of a drug:

Let

A_0 = starting amount (read: "A sub zero" or "A naught")
A = ending amount
e = a constant whose value is **approximately 2.718**.
k = growth or decay rate (expressed as a decimal)
t = time

Then

$$\boxed{A = A_0 e^{kt}}$$

This formula assumes **continuous** growth or decay, which means at "every moment in time." Don't be dismayed if that is hard to fathom.

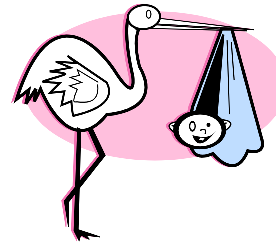
Get your calculator out. Here's a quick review of the "exponent" button. To calculate $(3.2)^{4.15}$, try either

3.2 $\boxed{y^x}$ 4.15 $\boxed{=}$ OR 3.2 $\boxed{\wedge}$ 4.15 $\boxed{=}$

The answer is about 124.845. [Your calculator may use $\boxed{\text{enter}}$ instead of the equals sign.]

The number ***e*** used in the Growth Formula is one of the most important numbers in math, statistics, science, engineering, and business. Like the number π, this real number contains an infinite number of digits which never form a repeating pattern (e and π and $\sqrt{2}$ are called *irrational* numbers).

EXAMPLE 1: **Assuming an initial population of 1,506 and a growth rate of 25% per year compounded continuously, predict the population in 10 years.**



Solution: Let's start by writing the formula that will solve this problem:

$$A = A_0 e^{kt}$$

The initial population is 1,506; so $A_0 = 1{,}506$.

The growth rate is 25%; thus $k = 0.25$.

We're talking about a period of 10 years; therefore $t = 10$.

Plug all these values into our formula (using 2.718 for e):

$$A = 1{,}506(2.718)^{(0.25)(10)}$$

$$\Rightarrow \quad A = 1{,}506(2.718)^{2.5}$$

$$\Rightarrow \quad A = 1{,}506(12.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 18{,}343}$$

EXAMPLE 2: **Assuming a starting investment of $2,401 and an annual interest rate of 13% compounded continuously, predict the value of the investment in 11 years.**

Solution: The formula we used for population growth works equally well for compound interest.

$$A = A_0 e^{kt}$$

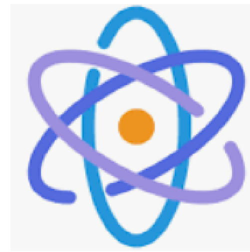
$$\Rightarrow \quad A = 2{,}401(2.718)^{(0.13)(11)}$$

$$\Rightarrow \quad A = 2{,}401(2.718)^{1.43}$$

$$\Rightarrow \quad A = 2{,}401(4.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = \$10{,}036}$$

<u>EXAMPLE 3:</u> **Starting with 624 grams of uranium and assuming an annual <u>decay</u> rate of 5%, compute the number of grams of uranium remaining after 9 years.**



<u>Solution:</u> We use the same growth/decay formula; but since the amount of uranium is shrinking (decaying) rather than growing, we will use a decay rate of **–5%** (that's <u>negative</u> 5 percent) in our formula:

$$A = A_0e^{kt}$$

$$\Rightarrow \quad A = 624(2.718)^{(-0.05)(9)}$$

$$\Rightarrow \quad A = 624(2.718)^{-0.45}$$

$$\Rightarrow \quad A = 624(0.64) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 399 \text{ grams}}$$

Homework

<u>Note:</u> The result of raising 2.718 to the exponent is assumed to be rounded to 2 digits, as in the examples.

1. Starting with 178 grams of uranium and assuming an annual decay rate of 17%, predict the number of grams remaining after 18 years.

2. Assuming an initial population of 3,902 and a growth rate of 14% per year, determine the population in 10 years.

3. Assuming an initial investment of $1,299 and an annual interest rate of 14% compounded continuously, predict the value of the investment in 4 years.

4. Assuming an initial investment of $9,574 and an annual interest rate of 11% compounded continuously, compute the value of the investment in 2 years.

5. Starting with 416 grams of thorium and assuming an annual decay rate of 3%, determine the number of grams remaining after 26 years.

6. Assuming an initial population of 2,586 and a growth rate of 16% per year, predict the population in 4 years.

7. Assuming an initial population of 1,422 and a growth rate of 12% per year, compute the population in 2 years.

8. Assuming an initial investment of $6,897 and an annual interest rate of 14% compounded continuously, compute the value of the investment in 6 years.

9. Starting with 215 grams of plutonium and assuming an annual decay rate of 15%, calculate the number of grams remaining after 26 years.

10. Assuming an initial population of 2,159 and a growth rate of 6% per year, calculate the population in 11 years.

[As mentioned on the second page of this Chapter, technically speaking, the growth/decay formula we've been using applies only when the growth or decay is *continuous*, which means the growth or decay occurs at every single moment. This may not strictly be the case in all situations (for example, in the births of people), but let's not worry about it in this chapter. Let's just use the formula to solve the problems.]

Solutions

1. 9 g	**2.** 15,803	**3.** $2,273	**4.** $11,968
5. 191 g	**6.** 4,913	**7.** 1,806	**8.** $16,001
9. 4 g	**10.** 4,167		

"Education is not the
filling of a pail,
but the lighting of a fire."

– *William Butler Yeats*